

Fredholm Solvability of Dissipative Hyperbolic Problems with Periodic Boundary Conditions

I. Kmit L. Recke

Institute of Mathematics, Humboldt University of Berlin,
 Rudower Chaussee 25, D-12489 Berlin, Germany
 and Institute for Applied Problems of Mechanics and Mathematics,
 Ukrainian Academy of Sciences, Naukova St. 3b, 79060 Lviv, Ukraine
 E-mail: kmit@informatik.hu-berlin.de

Institute of Mathematics, Humboldt University of Berlin,
 Rudower Chaussee 25, D-12489 Berlin, Germany
 E-mail: recke@mathematik.hu-berlin.de

Abstract

This paper concerns a linear first-order hyperbolic system in one space dimension of the type

$$\partial_t u_j + a_j(x, t) \partial_x u_j + \sum_{k=1}^n b_{jk}(x, t) u_k = f_j(x, t), \quad 1 \leq j \leq n,$$

with periodicity conditions in time and reflection boundary conditions in space. We state conditions on the coefficients a_j and b_{jk} and the reflection coefficients such that the system is Fredholm solvable in the space of continuous functions. In particular, these conditions exclude the small denominators effect.

Our results cover non-strictly hyperbolic systems, but they are new even in the case of strict hyperbolicity.

Key words: first-order hyperbolic systems, periodic-Dirichlet problems, small denominators, Fredholm solvability.

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1 Problem and main results

This paper concerns continuous solutions to general linear first-order hyperbolic systems in one space dimension of the type

$$\partial_t u_j + a_j(x, t) \partial_x u_j + \sum_{k=1}^n b_{jk}(x, t) u_k = f_j(x, t), \quad 1 \leq j \leq n, \quad (x, t) \in (0, 1) \times \mathbb{R} \quad (1)$$

with time-periodicity conditions

$$u_j(x, t + 2\pi) = u_j(x, t), \quad 1 \leq j \leq n, \quad x \in [0, 1] \quad (2)$$

and reflection boundary conditions

$$\begin{aligned} u_j(0, t) &= \sum_{k=m+1}^n r_{jk}^0 u_k(0, t), \quad 1 \leq j \leq m, \quad t \in \mathbb{R} \\ u_j(1, t) &= \sum_{k=1}^m r_{jk}^1 u_k(1, t), \quad m+1 \leq j \leq n, \quad t \in \mathbb{R}. \end{aligned} \quad (3)$$

Here the dimensions $0 \leq m < n$ and the reflection coefficients $r_{jk}^0, r_{jk}^1 \in \mathbb{R}$ are fixed. The right-hand sides f_j as well as the coefficient functions a_j and b_{jk} are supposed to be continuous and 2π -periodic with respect to t . We assume that

$$a_j(x, t) \neq 0 \quad \text{for all } 1 \leq j \leq n \quad \text{and } (x, t) \in [0, 1] \times \mathbb{R}. \quad (4)$$

Roughly speaking, we will prove a result of the following type: We will state conditions on the data a_j, b_{jk} , and r_{jk} such that the system (1)–(3) is Fredholm solvable, i.e.

- either the system (1)–(3) with $f = 0$ has a nontrivial solution (then the vector space of all solutions has finite positive dimension)
- or the system (1)–(3) has a unique solution for any right hand side f .

Here and in what follows by

$$a = \text{diag}(a_1, \dots, a_n), \quad f = (f_1, \dots, f_n), \quad u = (u_1, \dots, u_n), \quad \text{and } b = [b_{jk}]_{j,k=1}^n$$

we denote the diagonal matrix of the coefficient functions a_j , the vectors of the right hand sides f_j and the solutions u_j , and the matrix of the coefficient functions b_{jk} , respectively.

In order to formulate our results more precisely, let us introduce some function spaces: By $\mathcal{C}(\mathbb{R})$ we denote the space of all continuous functions $\varphi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi(x, t + 2\pi) = \varphi(x, t)$ for all $x \in [0, 1]$ and $t \in \mathbb{R}$, with the norm

$$\|\varphi\|_{\mathcal{C}(\mathbb{R})} = \max_{0 \leq x \leq 1} \max_{0 \leq t \leq 2\pi} |\varphi(x, t)|.$$

Further, by $\mathcal{C}(\mathbb{R}^n)$ and $\mathcal{C}(\mathbb{M}_n)$ we denote the vector spaces of all maps $u = (u_1, \dots, u_n) : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $b = [b_{jk}]_{j,k=1}^n : [0, 1] \times \mathbb{R} \rightarrow \mathbb{M}_n$ (\mathbb{M}_n is the vector space of all $n \times n$ matrices) such that $u_j \in \mathcal{C}(\mathbb{R})$ for all $1 \leq j \leq n$ and $b_{jk} \in \mathcal{C}(\mathbb{R})$ for all $1 \leq j, k \leq n$, with the norms

$$\|u\|_{\mathcal{C}(\mathbb{R}^n)} = \max_{1 \leq j \leq n} \|u_j\|_{\mathcal{C}(\mathbb{R})} \quad \text{and} \quad \|b\|_{\mathcal{C}(\mathbb{M}_n)} = \max_{1 \leq j, k \leq n} \|b_{jk}\|_{\mathcal{C}(\mathbb{R})},$$

respectively. Finally, we denote by $\mathcal{C}^1(\mathbb{R}^n)$ and $\mathcal{C}^1(\mathbb{M}_n)$ the vector spaces of all $u \in \mathcal{C}(\mathbb{R}^n)$ and all $b \in \mathcal{C}(\mathbb{M}_n)$ such that the partial derivatives $\partial_t u$ and $\partial_t b$ exist and are continuous, with the norms, respectively,

$$\|u\|_{\mathcal{C}^1(\mathbb{R}^n)} = \|u\|_{\mathcal{C}(\mathbb{R}^n)} + \|\partial_t u\|_{\mathcal{C}(\mathbb{R}^n)} \quad \text{and} \quad \|b\|_{\mathcal{C}^1(\mathbb{M}_n)} = \|b\|_{\mathcal{C}(\mathbb{M}_n)} + \|\partial_t b\|_{\mathcal{C}(\mathbb{M}_n)}.$$

Further, let us introduce the characteristics of the hyperbolic system (1). For given $1 \leq j \leq n$, $(x, t) \in [0, 1] \times \mathbb{R}$, and $a \in \mathcal{C}^1(\mathbb{R}^n)$ with (4), they are defined as the solutions $\xi \in [0, 1] \mapsto \omega_j(\xi; x, t) \in \mathbb{R}$ of the initial value problems

$$\partial_\xi \omega_j(\xi; x, t) = \frac{1}{a_j(\xi, \omega_j(\xi; x, t))}, \quad \omega_j(x; x, t) = t. \quad (5)$$

Moreover, we denote

$$\begin{aligned} c_j(\xi; x, t) &= \exp \left\{ \int_x^\xi \left(\frac{b_{jj}}{a_j} \right) (\eta, \omega_j(\eta; x, t)) d\eta \right\}, \\ d_j(\xi; x, t) &= \frac{c_j(\xi; x, t)}{a_j(\xi, \omega_j(\xi; x, t))}, \\ R(a, b) &= \max_{m+1 \leq j \leq n} \max_{1 \leq k \leq m} \max_{x \in [0, 1]} \max_{t \in [0, 2\pi]} \left\{ \int_x^1 \left(\frac{b_{jj}}{a_j} \right) (\eta, \omega_j(\eta; x, t)) d\eta \right. \\ &\quad \left. + \int_1^0 \left(\frac{b_{kk}}{a_k} \right) (\eta, \omega_k(\eta; 1, \omega_j(1; x, t))) d\eta \right\}. \end{aligned}$$

Straightforward calculations show that a C^1 -map $u : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a solution to the periodic-Dirichlet problem (1)–(3) if and only if it is 2π -periodic with respect to t and satisfies the following system of integral equations

$$\begin{aligned} u_j(x, t) &= c_j(0; x, t) \sum_{k=m+1}^n r_{jk}^0 u_k(0, \omega_j(0; x, t)) \\ &\quad - \int_0^x d_j(\xi; x, t) \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi; x, t)) u_k(\xi, \omega_j(\xi; x, t)) d\xi \\ &\quad + \int_0^x d_j(\xi; x, t) f_j(\xi, \omega_j(\xi; x, t)) d\xi, \quad 1 \leq j \leq m, \end{aligned} \quad (6)$$

$$\begin{aligned} u_j(x, t) &= c_j(1; x, t) \sum_{k=1}^m r_{jk}^1 u_k(1, \omega_j(1; x, t)) \\ &\quad + \int_x^1 d_j(\xi; x, t) \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi; x, t)) u_k(\xi, \omega_j(\xi; x, t)) d\xi \\ &\quad - \int_x^1 d_j(\xi; x, t) f_j(\xi, \omega_j(\xi; x, t)) d\xi, \quad m+1 \leq j \leq n. \end{aligned} \quad (7)$$

This motivates the following definition:

Definition 1.1 *A function $u \in \mathcal{C}(\mathbb{R}^n)$ is called a continuous solution to (1)–(3) if it satisfies (6) and (7).*

The vector space of all continuous solutions to the homogeneous problem (1)–(3) (i.e. (1)–(3) with $f_j = 0$ for all $1 \leq j \leq n$) will be denoted by \mathcal{K} .

Our first result concerns the Fredholm solvability of (1)–(3).

Theorem 1.2 *Let $a, b \in C^1(\mathbb{M}_n)$ satisfy (4) and the following conditions:*

$$R(a, b) \max_{j \leq n} \left\{ \exp \left\| 2\partial_t a_j / a_j^2 \right\|_{C(\mathbb{R})} \right\} \sum_{j,l=m+1}^n \sum_{k=1}^m |r_{jk}^1 r_{kl}^0| < 1, \quad (8)$$

$$\begin{aligned} & \text{for all } j \neq k \text{ the set } \{(x, t) \in [0, 1] \times [0, 2\pi] : a_k - a_j = 0\} \\ & \text{consists of a finite number of connected components,} \end{aligned} \quad (9)$$

and

$$\text{for all } j \neq k \text{ there exists } p_{jk} \in C^1([0, 1] \times \mathbb{R}) \text{ with } b_{jk} a_k = p_{jk}(a_k - a_j). \quad (10)$$

Then the following is true:

- (i) $\dim \mathcal{K} < \infty$.
- (ii) Either $\dim \mathcal{K} > 0$ or for any $f \in C(\mathbb{R}^n)$ there exists exactly one continuous solution to (1)–(3).
- (iii) If $b_{jk} = 0$ for all $1 \leq j \neq k \leq n$, then $\dim \mathcal{K} = 0$.

In order to characterize more explicitly the set of all right hand sides f such that (1)–(3) has a continuous solution, we consider the following formally adjoint system to the homogeneous variant of (1)–(3):

$$-\partial_t u_j - \partial_x (a_j(x, t) u_j) + \sum_{k=1}^n b_{kj}(x, t) u_k = 0, \quad 1 \leq j \leq n, \quad x \in (0, 1), \quad t \in \mathbb{R}, \quad (11)$$

$$u_j(x, t + 2\pi) = u_j(x, t), \quad 1 \leq j \leq n, \quad x \in [0, 1], \quad t \in \mathbb{R}, \quad (12)$$

$$\begin{aligned} a_j(0, t) u_j(0, t) &= - \sum_{k=1}^m r_{kj}^0 a_k(0, t) u_k(0, t), \quad m+1 \leq j \leq n, \quad t \in \mathbb{R}, \\ a_j(1, t) u_j(1, t) &= - \sum_{k=m+1}^n r_{kj}^1 a_k(1, t) u_k(1, t), \quad 1 \leq j \leq m, \quad t \in \mathbb{R}. \end{aligned} \quad (13)$$

Further, we write

$$\langle f, u \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \langle f(x, t), u(x, t) \rangle \, dx dt$$

for the usual scalar product in the Hilbert space $L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n)$, and $\langle \cdot, \cdot \rangle$ for the Euclidean scalar product in \mathbb{R}^n . Moreover, the vector space of all continuous solutions to the problem (11)–(13) will be denoted by $\tilde{\mathcal{K}}$.

We are now prepared to formulate our second result.

Theorem 1.3 *Let $a \in C^1([0, 1] \times [0, 2\pi]; \mathbb{M}_n)$, $b \in C^1(\mathbb{M}_n)$ be given such that (4) and (8)–(10) are fulfilled. Suppose also that*

$$\begin{aligned} & R(a, \partial_x a - b) \max_{j \leq n} \left\{ \exp \left\| 2\partial_t a_j / a_j^2 \right\|_{C(\mathbb{R})} \right\} \\ & \times \sum_{j,l=m+1}^n \sum_{k=1}^m \max_{x,t} \left| \left(\frac{a_k}{a_j} \right) (1, \omega_j(1; x, t)) \left(\frac{a_l}{a_k} \right) (0, \omega_k(0; 1, \omega_j(1; x, t))) \right| |r_{kj}^1 r_{lk}^0| < 1. \end{aligned} \quad (14)$$

Then the following is true:

- (i) $\dim \mathcal{K} = \dim \tilde{\mathcal{K}}$.
- (ii) Given $f \in \mathcal{C}(\mathbb{R}^n)$, there exists a continuous solution to (1)–(3) if and only if $\langle f, u \rangle_{L^2} = 0$ for all $u \in \tilde{\mathcal{K}}$.

Remark 1.4 Assumption (10) is not necessary, but cannot be avoided in general: Since the set of Fredholm operators is open, the conclusion of Theorem 1.2 survives under small (in $\mathcal{C}^1(\mathbb{R})$) perturbations of the coefficients b_{jk} . Hence the condition (10) is not necessary for the conclusions of Theorems 1.2 and 1.3. From the other side, the condition (10) can not be avoided in general. Indeed, the following example shows that, in the case of multiple characteristics, we lose the Fredholmness property if the lower-order terms coefficients are large enough. Specifically, we consider a particular case of (1)–(3):

$$\begin{aligned} \partial_t u_1 + \partial_x u_1 &= \partial_t u_2 + \partial_x u_2 + b u_1 &= 0, \\ u_1(x, t + 2\pi) - u_1(x, t) &= u_2(x, t + 2\pi) - u_2(x, t) &= 0, \\ u_1(0, t) - r_{12}^0 u_2(0, t) &= u_2(1, t) - r_{21}^1 u_1(1, t) &= 0, \end{aligned}$$

where b is a constant. If $r_{12}^0 r_{21}^1 < 1$ (no small denominators occur) and $b \neq 0$, then all assumptions of Theorems 1.2 and 1.3 are fulfilled with the exception of (10). If, moreover,

$$b = \frac{r_{12}^0 r_{21}^1 - 1}{r_{12}^0}$$

(covering large enough $|b|$), then

$$u_1(x, t) = \sin l(t - x), \quad u_2(x, t) = b \left(\frac{1}{1 - r_{12}^0 r_{21}^1} - x \right) \sin l(t - x), \quad l \in \mathbb{N},$$

are infinitely many linearly independent solutions. Hence, the Fredholmness is destroyed.

The present paper has been motivated mainly by two reasons:

From the theoretical point of view, we are interested in developing a bifurcation theory for hyperbolic PDEs. The basic idea is to apply techniques based on the Implicit Function Theorem in Banach spaces and the Lyapunov-Schmidt reduction (see, e.g., [1, 8]). Note that the Fredholm solvability of the linearization is the key point here.

From the practical point of view, some problems of mathematical biology [4, 5, 6, 15], kinetic gas dynamics [2, 7, 17], and semiconductor laser dynamics [13, 18, 19] are covered by (1)–(3).

This paper develops the ideas presented in [9, 10, 11].

2 Fredholm alternative (proof of Theorem 1.2)

We start with writing the problem (6)–(7) in an operator form. Along with the notation $\omega_j(\xi; x, t)$, we will also use its shortened version $\omega_j(\xi)$. Moreover, we introduce vectors

$$u' = (u_1, \dots, u_m), \quad u'' = (u_{m+1}, \dots, u_n)$$

and linear operators $C_1, E_1 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^m))$, $C_2, E_2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))$, $(B_1, B_2) \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$, $R_1 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}), \mathcal{C}(\mathbb{R}^m))$, $R_2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^m), \mathcal{C}(\mathbb{R}^{n-m}))$ by

$$\begin{aligned}
(C_1 u')(x, t) &= (c_1(0; x, t)u_1(x, t), \dots, c_m(0; x, t)u_m(x, t)) \\
(C_2 u'')(x, t) &= (c_{m+1}(1; x, t)u_{m+1}(x, t), \dots, c_n(1; x, t)u_n(x, t)) \\
(R_1 u'')(x, t) &= \left[\sum_{k=m+1}^n r_{jk}^0 u_k(0, \omega_j(0)) \right]_{j=1}^m, \\
(R_2 u')(x, t) &= \left[\sum_{k=1}^m r_{jk}^1 u_k(1, \omega_j(1)) \right]_{j=m+1}^n, \\
(B_1 u)(x, t) &= \left[\sum_{k \neq j} b_{jk} u_k \right]_{j=1}^m, \\
(B_2 u)(x, t) &= \left[\sum_{k \neq j} b_{jk} u_k \right]_{j=m+1}^n, \\
(E_1 u')(x, t) &= \left[\int_0^x d_j(\xi; x, t) u_j(\xi, \omega_j(\xi)) d\xi \right]_{j=1}^m, \\
(E_2 u'')(x, t) &= \left[\int_x^1 d_j(\xi; x, t) u_j(\xi, \omega_j(\xi)) d\xi \right]_{j=m+1}^n.
\end{aligned}$$

In these notations the problem (6)–(7) admits the following representation:

$$\begin{aligned}
u' &= C_1 R_1 u'' - E_1 B_1 u + E_1 f', \\
u'' &= C_2 R_2 u' + E_2 B_2 u - E_2 f''
\end{aligned} \tag{2.1}$$

or, the same,

$$\begin{aligned}
u' &= C_1 R_1 u'' - E_1 B_1 u + E_1 f', \\
u'' &= C_2 R_2 C_1 R_1 u'' - C_2 R_2 E_1 B_1 u + E_2 B_2 u + C_2 R_2 E_1 f' - E_2 f''.
\end{aligned} \tag{2.2}$$

Write

$$\begin{aligned}
Mu &= (M_1 u'', M_2 u'') = (C_1 R_1 u'', C_2 R_2 C_1 R_1 u''), \\
Ku &= (K_1 u, K_2 u) = (-E_1 B_1 u, E_2 B_2 u - C_2 R_2 E_1 B_1 u), \\
Ff &= (E_1 f', C_2 R_2 E_1 f' - E_2 f'')
\end{aligned}$$

for the linear operators $M, K, F \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$. Then the operator equation

$$u = Mu + Ku + Ff \tag{2.3}$$

gives an abstract representation of the problem (6)–(7).

2.1 Fredholmness criterion

Theorem 2.1 *Let W be a Banach space, I the identity in W , and $K \in \mathcal{L}(W)$ with K^2 being compact. Then $I + K$ is a Fredholm operator of index zero.*

Proof. Since

$$I - K^2 = (I + K)(I - K) = (I - K)(I + K),$$

the operator $I - K$ is a parametrix for the operator $I + K \in \mathcal{L}(W)$. This implies the Fredholmness of $I + K$ by [21, Proposition 5.7.1] or [20, Theorem 5.5]. Nevertheless, for the reader's convenience here we give an independent, simple, and self-contained proof of this fact, see also [10]. Note first that

$$\dim \ker(I + K) \leq \dim \ker(I - K^2) < \infty. \quad (2.4)$$

Similarly $\dim \ker(I + K)^* < \infty$, hence $\overline{\operatorname{im}(I + K)} < \infty$. It remains to show that $\operatorname{im}(I + K)$ is closed.

Take a sequence $(w_j) \subset W$ and an element $w \in W$ such that

$$(I + K)w_j \rightarrow w. \quad (2.5)$$

We have to show that $w \in \operatorname{im}(I + K)$.

By (2.4) there exists a closed subspace V of W such that

$$W = \ker(I + K) \oplus V, \quad (2.6)$$

Consider the decomposition

$$w_j = u_j + v_j, \text{ where } u_j \in \ker(I + K) \text{ and } v_j \in V.$$

From (2.5) we infer that

$$(I + K)v_j \rightarrow w. \quad (2.7)$$

Let us show that the sequence (v_j) is bounded. Suppose this is not true. Without loss of generality we can assume that

$$\lim_{j \rightarrow \infty} \|v_j\| = \infty. \quad (2.8)$$

From (2.7) and (2.8) we get

$$(I + K) \frac{v_j}{\|v_j\|} \rightarrow 0, \quad (2.9)$$

hence

$$(I - K^2) \frac{v_j}{\|v_j\|} \rightarrow 0. \quad (2.10)$$

Since K^2 is compact, there exist $v \in W$ and a subsequence $(v_{j_k})_{k \in \mathbb{N}}$ such that

$$K^2 \frac{v_{j_k}}{\|v_{j_k}\|} \rightarrow v. \quad (2.11)$$

The convergences (2.11) and (2.10) immediately imply that

$$\frac{v_{j_k}}{\|v_{j_k}\|} \rightarrow v \in V. \quad (2.12)$$

Combining (2.12) with (2.9), we get $(I + K)v = 0$, i.e. $v \in V \cap \ker(I + K)$ and $\|v\| = 1$. This contradicts (2.6) and proves the boundedness of (v_j) .

Now we show that $w \in \text{im}(I + K)$. As K^2 is compact, there exists $v \in W$ and a subsequence (v_{j_k}) such that $K^2 v_{j_k} \rightarrow v$ as $k \rightarrow \infty$. By (2.7) we also have

$$(I - K^2)v_j = (I - K)(I + K)v_j \rightarrow (I - K)w.$$

Therefore,

$$\lim_{k \rightarrow \infty} v_{j_k} = (I - K)w + v$$

and

$$w = \lim_{k \rightarrow \infty} (I + K)v_{j_k} = (I + K)((I - K)w + v) \in \text{im}(I + K)$$

as desired. The Fredholm property is thereby proved.

To prove that $I + K$ has index zero, we additionally use a homotopy argument. Let us consider the continuous function

$$s \in \mathbb{R} \mapsto I + sK \in \mathcal{L}(W).$$

Since $K^2 \in \mathcal{L}(W)$ is a compact operator, the operators $(sK)^2 \in \mathcal{L}(W)$ are compact for each $s \in \mathbb{R}$ and, as we just proved, the operators $I + sK$ are Fredholm. By [21, Proposition 5.8.1], $\text{ind}(I + sK) = \text{const}$ for all $s \in \mathbb{R}$. It remains to notice that the identity operator I has index zero. \square

Our aim is to prove that $I - M - K \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ is a Fredholm operator of index zero. In Section 2.2 we show that $I - M \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ is bijective. Hence the Fredholmness of $I - M - K \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ is equivalent to the Fredholmness of $I - (I - M)^{-1}K \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$. By Theorem 2.1, we are done if we prove the compactness of $((I - M)^{-1}K)^2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ or, the same, the compactness of $K(I - M)^{-1}K \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$. The latter is the subject of Section 2.3.

2.2 Isomorphism property

The equivalence of the systems (2.1) and (2.2) implies that the bijectivity of $I - M \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ is equivalent to the bijectivity of $I - M_2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))$. Since

$$\|M_2\|_{\mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))} = \|C_2 R_2 C_1 R_1\|_{\mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))} \leq R(a, b) \sum_{j,l=m+1}^n \sum_{k=1}^m |r_{jk}^1 r_{kl}^0|,$$

the assumption (8) means the contractibility of M_2 . Now, the bijectivity of $I - M_2$ and, hence, the bijectivity of $I - M$ follows directly from the Banach fixed point theorem.

2.3 Fredholm property

In this section we prove the compactness of $K(I - M)^{-1}K \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$. Since $I - M_2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))$ is invertible, the operator $(I - M)^{-1} \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ is given by

$$(I - M)^{-1}u = (M_1(I - M_2)^{-1}u'' + u', (I - M_2)^{-1}u'').$$

This entails that

$$K(I - M)^{-1}K = (K_1, K_2) (M_1(I - M_2)^{-1}K_2 + K_1, (I - M_2)^{-1}K_2).$$

Obviously, the compactness of $K(I - M)^{-1}K$ will follow from the compactness of the operators

$$K_i (M_1(I - M_2)^{-1}K_2 + K_1, (I - M_2)^{-1}K_2), \quad i = 1, 2,$$

or from the compactness of the following four operators:

$$E_i B_i (M_1(I - M_2)^{-1}K_2, (I - M_2)^{-1}K_2), \quad i = 1, 2,$$

and

$$E_i B_i (K_1, (I - M_2)^{-1}K_2), \quad i = 1, 2.$$

The latter follows from the definitions of K_1 and K_2 and the fact that, if $L \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n), \mathcal{C}(\mathbb{R}^m))$ is a compact operator, then the operator $C_2 R_2 L \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n), \mathcal{C}(\mathbb{R}^{n-m}))$ is compact as well. We will treat only one of the operators, say,

$$E_1 B_1 (K_1, (I - M_2)^{-1}K_2)$$

(the other three can be treated similarly). Making further simplification, we use the definitions of K_1 and K_2 again and see that it is sufficient to prove the compactness of the operators

$$E_1 B_1 (E_1 B_1, (I - M_2)^{-1}E_2 B_2)$$

and

$$E_1 B_1 (E_1 B_1, (I - M_2)^{-1}C_2 R_2 E_1 B_1),$$

which can again be handled by a similar argument. We will show that the operator

$$E_1 B_1 (E_1 B_1, (I - M_2)^{-1}E_2 B_2)$$

is compact. We actually have to prove the precompactness of the set

$$\{Au : u \in N\},$$

where

$$Au = E_1 B_1 (E_1 u', (I - M_2)^{-1}E_2 u'')$$

and $N \subset \mathcal{C}(\mathbb{R}^n)$ is an arbitrary fixed bounded set. With this aim we will use the Arzela-Ascoli precompactness criterion in $\mathcal{C}(\mathbb{R}^n)$. As $A \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ is a bounded operator, the set AN is bounded in $\mathcal{C}(\mathbb{R}^n)$. Therefore, our task is reduced to check the equicontinuity

property of AN in $\mathcal{C}(\mathbb{R}^n)$. We are done if we show the existence of a function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}$ with $\alpha(p) \rightarrow 0$ as $p \rightarrow 0$ for which we have

$$\|(Au)(x + h_1, t + h_2) - (Au)(x, t)\|_{\mathcal{C}(\mathbb{R}^n)} \leq \alpha(|h|), \quad (2.13)$$

the estimate being uniform in $u \in N$ and $(h_1, h_2) \in \mathbb{R}^2$. Here $|h| = |h_1| + |h_2|$. Note that the equicontinuity property (2.13) is not straightforward because u under the integrals in the representation of $(Au)(x, t)$ depends not only on the variables of integration, but also on the free variables x and t .

Using the continuous and dense embedding

$$C^1([0, 1] \times \mathbb{R}) \hookrightarrow C([0, 1] \times \mathbb{R}),$$

given $\varepsilon > 0$, for each $u \in N$ we fix a sequence $u^l \in C^1([0, 1] \times \mathbb{R})$ such that

$$u^l \rightarrow u \text{ in } \mathcal{C}(\mathbb{R}^n) \quad \text{and} \quad \|u^l - u\|_{\mathcal{C}(\mathbb{R}^n)} < \varepsilon. \quad (2.14)$$

Hence

$$(Au)(x, t) = \lim_{l \rightarrow \infty} (Au^l)(x, t) \text{ in } \mathcal{C}(\mathbb{R}^n) \quad \text{and} \quad \|(Au^l)(x, t) - (Au)(x, t)\|_{\mathcal{C}(\mathbb{R}^n)} < \varepsilon \|A\|_{\mathcal{L}(\mathcal{C}(\mathbb{R}^n))}. \quad (2.15)$$

Note that $(Au^l)(x, t) \in C^1([0, 1] \times \mathbb{R})$ for each $l \in \mathbb{N}$. We are done if we show that the gradient $\nabla_{(x,t)} [(Au^l)(x, t)]$ is bounded by a constant $C > 0$, uniformly in $u \in N$, $(x, t) \in [0, 1] \times [0, 2\pi]$, and $l \in \mathbb{N}$. Indeed, then we have the bound

$$\begin{aligned} & \|(Au)(x + h_1, t + h_2) - (Au)(x, t)\|_{\mathcal{C}(\mathbb{R}^n)} \\ & \leq \|(Au)(x + h_1, t + h_2) - (Au^l)(x + h_1, t + h_2)\|_{\mathcal{C}(\mathbb{R}^n)} \\ & \quad + \|(Au^l)(x + h_1, t + h_2) - (Au^l)(x, t)\|_{\mathcal{C}(\mathbb{R}^n)} \\ & \quad + |h| \left\| \int_0^1 \int_0^1 \nabla [(Au^l)(\sigma_1(x + h_1) + (1 - \sigma_1)x, \sigma_2(t + h_2) + (1 - \sigma_2)t)] d\sigma_1 d\sigma_2 \right\|_{\mathcal{C}(\mathbb{R}^n)} \\ & \leq 2\varepsilon \|A\|_{\mathcal{L}(\mathcal{C}(\mathbb{R}^n))} + |h|C. \end{aligned} \quad (2.16)$$

It remains to note that this estimate is true for any sufficiently small $\varepsilon > 0$.

Now we prove the uniform boundedness of $\nabla_{(x,t)} [(Au^l)(x, t)]$. To simplify technicalities, we take into account the definition of B_1 and rewrite Au in the form:

$$Au = \left[\sum_{\substack{k=1 \\ k \neq j}}^m (E_1)_j b_{jk} (E_1)_k u' + \sum_{k=m+1}^n (E_1)_j b_{jk} ((I - M_2)^{-1} E_2)_k u'' \right]_{j=1}^m,$$

where $(Q)_j$ denotes the j -th component of the operator Q . Therefore, it is sufficient to show the uniform boundedness of the gradient of the integral expressions of kinds

$(E_1)_j b_{jk} (E_1)_k u'$ and $(E_1)_j b_{jk} ((I - M_2)^{-1} E_2)_k u''$. We will concentrate only on the last expression. The desired property for the former one will then easily follow from the estimates obtained below for the first term ($q = 0$) of the series (2.17).

Since $\|M_2\|_{\mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))} < 1$, the inverse to $I - M_2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))$ is representable by the Neumann series

$$(I - M_2)^{-1} = \sum_{q=0}^{\infty} M_2^q,$$

and we will be concerned with the expression

$$(E_1)_j b_{jk} \left(\sum_{q=0}^{\infty} M_2^q E_2 \right)_k u'' = \sum_{q=0}^{\infty} (E_1)_j b_{jk} (M_2^q E_2)_k u'' = \sum_{q=0}^{\infty} (E_1)_j b_{jk} ((C_2 R_2 C_1 R_1)^q E_2)_k u'' \quad (2.17)$$

for arbitrarily fixed $1 \leq j \leq m$ and $m+1 \leq k \leq n$.

Let us outline the proof. We will treat each term in the series (2.17) (with $u = u^l$) separately, each time following the same argument and at the same time taking care about the q -dependence in all subsequent estimates. Given $q \in \mathbb{N}_0$, we take into account that the q -th term is a finite sum of integral expressions (due to the definitions of R_1 and R_2) that can be handled similarly. Hence, to treat the q -th term, it suffices to derive the desired estimate for one arbitrarily fixed summand contributing into this term. Then, summing up all the estimates in $q \in \mathbb{N}_0$ and using the contractibility condition (8), we come to the final estimate. It should be noted here that the continuous differentiability of the series (2.17) with u^l in place of u follows from the termwise differentiability of (2.17) and the uniform convergency of the series of the derivatives (see the estimates below).

Note first that, given $a_j \in C_t^1([0, 1] \times \mathbb{R})$, the functions $\omega_j(\xi; x, t)$ for all $1 \leq j \leq n$ are C^1 in their arguments and the respective derivatives are given by (5) and

$$\begin{aligned} \frac{\partial \omega_j(\xi; x, t)}{\partial x} &= -a_j^{-1}(x, t) \exp \left\{ \int_x^\xi \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta \right\}, \\ \frac{\partial \omega_j(\xi; x, t)}{\partial t} &= \exp \left\{ \int_x^\xi \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta; x, t)) d\eta \right\}, \end{aligned} \quad (2.18)$$

where ∂_m here and in what follows denotes the derivative with respect to the m -th argument.

Running our argument, we start with $q = 0$, namely with the integral expression

$$A_0(x, t) = \int_0^x d_j(\xi; x, t) b_{jk}(\xi, \omega_j(\xi)) \int_\xi^1 d_k(\xi_1; \xi, \omega_j(\xi)) u_k^l(\xi_1, \omega_k(\xi_1; \xi, \omega_j(\xi))) d\xi_1 d\xi. \quad (2.19)$$

The subscript in A_0 corresponds to $q = 0$. To simplify notation, we drop the dependence of A_0 on j, k , and u_k^l . Now we derive an upper bound for $\partial_x A_0(x, t)$ (a similar bound for $\partial_t A_0(x, t)$ then will easily follow from (2.54)).

On the account of (2.18),

$$\partial_x A_0(x, t) = A_0^1(x, t) + A_0^2(x, t) + A_0^3(x, t) + A_0^4(x, t), \quad (2.20)$$

where

$$\begin{aligned}
A_0^1(x, t) &= \frac{b_{jk}(x, t)}{a_j(x, t)} \int_x^1 d_k(\xi; x, t) u_k^l(\xi, \omega_k(\xi)) d\xi, \\
A_0^2(x, t) &= \int_0^x \int_\xi^1 \partial_x [d_j(\xi; x, t) b_{jk}(\xi, \omega_j(\xi)) d_k(\xi_1; \xi, \omega_j(\xi))] u_k^l(\xi_1, \omega_k(\xi_1; \xi, \omega_j(\xi))) d\xi_1 d\xi, \\
A_0^3(x, t) &= 0, \\
A_0^4(x, t) &= -\frac{1}{a_j(x, t)} \int_0^x \int_\xi^1 d_j(\xi; x, t) b_{jk}(\xi, \omega_j(\xi)) d_k(\xi_1; \xi, \omega_j(\xi)) \\
&\quad \times \exp \left\{ \int_x^\xi \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \exp \left\{ \int_\xi^{\xi_1} \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; \xi, \omega_j(\xi))) d\eta \right\} \\
&\quad \times (\partial_2 u_k^l)(\xi_1, \omega_k(\xi_1; \xi, \omega_j(\xi))) d\xi_1 d\xi.
\end{aligned}$$

Note that the x -derivative of the q -th term for $q \geq 1$, denoted below by $\partial_x A_q(x, t)$, is again representable by the sum of four summands of these four kinds only. Remark that for each $q \in \mathbb{N}_0$ the third summand, denoted by $A_q^3(x, t)$, is responsible for the part of the x -derivative that contains derivatives of functions c_j and comes from the operators C_1 and C_2 . Obviously, $A_0^3(x, t) \equiv 0$, but it is added to the right-hand side of (2.20) for the sake of completeness. Therefore, given $q \in \mathbb{N}_0$, we are reduced to estimate integrals of four kinds.

To perform all necessary estimates, we will use the following notation:

$$\begin{aligned}
L_{exp} &= \max_{j \leq n} \left\{ \exp \left\| \partial_t a_j / a_j^2 \right\|_{C(\mathbb{R})} \right\}, \quad L_p = \max_{1 \leq j \leq m} \max_{m+1 \leq k \leq n} \|p_{jk}\|_{C(\mathbb{R})}, \\
L_a &= \|a_j^{-1}\|_{C(\mathbb{R})}, \quad L'_a = \|\partial_t a_j^{-1}\|_{C(\mathbb{R})}, \quad L_b = \|b_{jk}\|_{C(\mathbb{R})}, \quad L'_b = \|\partial_t b_{jk}\|_{C(\mathbb{R})}, \\
L_d &= \max_{j \leq n} \max_{\xi \in [0, 1]} \|d_j(\xi; \cdot, \cdot)\|_{C(\mathbb{R})}, \quad L'_d = \sum_{i=2}^3 \max_{j \leq n} \max_{\xi \in [0, 1]} \|(\partial_i d_j)(\xi; \cdot, \cdot)\|_{C(\mathbb{R})}, \\
L_c &= \max_{j \leq n} \max_{\xi \in [0, 1]} \|c_j(\xi; \cdot, \cdot)\|_{C(\mathbb{R})}, \quad L'_c = \sum_{i=2}^3 \max_{j \leq n} \max_{\xi \in [0, 1]} \|(\partial_i c_j)(\xi; \cdot, \cdot)\|_{C(\mathbb{R})}.
\end{aligned} \tag{2.21}$$

Turning back to $q = 0$ and using (2.18), we get

$$\begin{aligned}
\|A_0^1(x, t)\|_{C(\mathbb{R})} &\leq L_a L_b L_d \|u_k^l\|_{C(\mathbb{R})}, \\
\|A_0^2(x, t)\|_{C(\mathbb{R})} &\leq L_d \left(\max_{\xi \in [0, 1]} \|\partial_x [d_j(\xi; \cdot, \cdot) b_{jk}(\xi, \omega_j(\xi; \cdot, \cdot))]\|_{C(\mathbb{R})} \right. \\
&\quad \left. + L_a L_b L'_d L_{exp} \right) \|u_k^l\|_{C(\mathbb{R})}.
\end{aligned}$$

To estimate $A_0^4(x, t)$, we transform the latter to the form where $\partial_2 u_k^l$ does not depend on x and t . Note first that, given $j \neq k$ and $(x, t) \in [0, 1] \times [0, 2\pi]$, the set $\{\tilde{x} \in [0, x] : (a_j - a_k)(\tilde{x}, \omega_j(\tilde{x})) \neq 0\}$ is a union of intervals, say $(y_i(x, t), z_i(x, t))$ or simply (y_i, z_i) .

Remark that the number of those intervals as a function of (x, t) , by assumption (9) is bounded from above uniformly on $[0, 1] \times [0, 2\pi]$ by a number of connected components of $[0, 1] \times [0, 2\pi]$ on which $a_k - a_j \neq 0$. The latter number will be denoted by N . Thanks to the assumption (10), we have

$$A_0^4(x, t) = \int_0^x \int_{\xi}^1 d\xi_1 d\xi = \sum_i \int_{y_i}^{z_i} \int_{\xi}^1 d\xi_1 d\xi.$$

We will transform each of the integrals in the sum separately, following the same scheme: For an arbitrarily fixed i we change variables $(\xi_1, \xi) \longrightarrow (\mu, \tau)$ by

$$\mu = \xi_1, \quad \tau = \omega_k(\xi_1; \xi, \omega_j(\xi)), \quad (2.22)$$

deriving therewith the formula

$$\begin{aligned} A_0^4(x, t) &= -a_j^{-1}(x, t) \sum_i \int_{S_i^0(x, t)} c_j(\rho; x, t) \left(\frac{b_{jk}}{a_j} \right) (\rho, \omega_j(\rho)) d_k(\mu; \rho, \omega_j(\rho)) \\ &\quad \times \exp \left\{ \int_x^\rho \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \exp \left\{ \int_\rho^\mu \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; \rho, \omega_j(\rho))) d\eta \right\} \\ &\quad \times |J_0(\mu, \rho)| \Big|_{\rho=\theta(\mu, \tau; x, t)} \partial_\tau u_s^l(\mu, \tau) d\mu d\tau, \end{aligned} \quad (2.23)$$

being true up to the sign (note the sign is not important for our analysis). Here $J_0(\mu, \rho)$ is the Jacobian of the transformation and $\theta(\mu, \tau; x, t)$ under the integral over $S_i^0(x, t)$ denotes the abscissa (in the range (y_i, z_i)) of the point where the characteristics $\omega_j(\rho; x, t)$ and $\omega_k(\rho; \mu, \tau)$ intersect, namely

$$\omega_j(\theta(\mu, \tau; x, t); x, t) = \omega_k(\theta(\mu, \tau; x, t); \mu, \tau). \quad (2.24)$$

Moreover, $S_i^0(x, t) \subset [0, 1] \times \mathbb{R}$ denotes the area bounded by the curves

$$\begin{aligned} &\{(\mu, \tau) \in [0, 1] \times [0, 2\pi] : \tau = \omega_j(\mu), y_i \leq \mu \leq z_i\}, \\ &\{(\mu, \tau) \in [0, 1] \times [0, 2\pi] : \tau = \omega_k(\mu; y_i, \omega_j(y_i)), y_i \leq \mu \leq 1\}, \\ &\{(\mu, \tau) \in [0, 1] \times [0, 2\pi] : \tau = \omega_k(\mu; z_i, \omega_j(z_i)), z_i \leq \mu \leq 1\}. \end{aligned} \quad (2.25)$$

Computing $J_0(\mu, \rho)$ precisely, we apply the formula $J_0(\mu, \rho) = \tilde{J}_0^{-1}(\xi_1, \xi)|_{(\xi_1, \xi)=(\mu, \rho)}$ with

$$\begin{aligned} \tilde{J}_0(\xi_1, \xi) &= \frac{\partial(\mu, \tau)}{\partial(\xi_1, \xi)} \\ &= \begin{vmatrix} 1 & 0 \\ (\partial_1 \omega_k)(\xi_1; \xi, \omega_j(\xi)) & (\partial_2 \omega_k)(\xi_1; \xi, \omega_j(\xi)) + (\partial_3 \omega_k)(\xi_1; \xi, \omega_j(\xi)) \omega_j'(\xi) \end{vmatrix}. \end{aligned}$$

On the account of (2.18),

$$\tilde{J}_0(\xi_1, \xi) = (a_j^{-1} - a_k^{-1})(\xi, \omega_j(\xi)) \exp \left\{ \int_\xi^{\xi_1} \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; \xi, \omega_j(\xi))) d\eta \right\}. \quad (2.26)$$

Hence,

$$J_0(\mu, \rho) = \left(\frac{a_k a_j}{a_k - a_j} \right) (\rho, \omega_j(\rho)) \exp \left\{ \int_{\mu}^{\rho} \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; \rho, \omega_j(\rho))) d\eta \right\}. \quad (2.27)$$

Due to the definition of (y_i, z_i) , the transformation of variables (2.22) is non-degenerate inside the domain of integration. This means that the inverse exists and is a C^1 -smooth function inside $S_i^0(x, t)$ for each $i \leq n$. Hence, $\theta(\mu, \tau; x, t)$ is C^1 in μ, τ inside $S_i^0(x, t)$ for each $(x, t) \in [0, 1] \times [0, 2\pi]$. Moreover, from the formulas (2.18) and (2.24) we have

$$\begin{aligned} \partial_{\mu} \theta(\mu, \tau; x, t) &= (\partial_2 \omega_k)(\rho; \mu, \tau) \left((\partial_1 \omega_j)(\rho; x, t) - (\partial_1 \omega_k)(\rho; \mu, \tau) \right)^{-1} \Big|_{\rho=\theta(\mu, \tau; x, t)} \\ &= -a_k^{-1}(\mu, \tau) \exp \left\{ \int_{\mu}^{\rho} \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; \mu, \tau)) d\eta \right\} \\ &\quad \times \left(\frac{1}{a_j(\rho, \omega_j(\rho; x, t))} - \frac{1}{a_k(\rho, \omega_k(\rho; \mu, \tau))} \right)^{-1} \Big|_{\rho=\theta(\mu, \tau; x, t)} \end{aligned} \quad (2.28)$$

and

$$\partial_{\tau} \theta(\mu, \tau; x, t) = -a_k(\mu, \tau) \partial_{\mu} \theta(\mu, \tau; x, t). \quad (2.29)$$

Note that, by the conditions (4), (9), and the construction of $S_i^0(x, t)$, given $i \leq N$, each of the functions $a_j(\rho, \omega_j(\rho))$, $a_k(\rho, \omega_j(\rho))$, and $\left(\frac{a_k}{a_k - a_j} \right) (\rho, \omega_j(\rho))$ on $S_i^0(x, t)$ keeps the same sign. Plugging (2.27) into (2.23), we arrive at the formula (being true up to the sign; all transformations of $A_0^4(x, t)$ below will be done up to the sign as well)

$$\begin{aligned} A_0^4(x, t) &= -a_j^{-1}(x, t) \sum_i \int_{S_i^0(x, t)} c_j(\rho; x, t) d_k(\mu; \rho, \omega_j(\rho)) \left(b_{jk} \left| \frac{a_k}{a_k - a_j} \right| \right) (\rho, \omega_j(\rho)) \\ &\quad \times \exp \left\{ \int_x^{\rho} \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \Big|_{\rho=\theta(\mu, \tau; x, t)} \partial_{\tau} u_k^l(\mu, \tau) d\mu d\tau \\ &= -a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \int_{S_i^0(x, t)} c_j(\rho; x, t) d_k(\mu; \rho, \omega_j(\rho)) p_{jk}(\rho, \omega_j(\rho)) \\ &\quad \times \exp \left\{ \int_x^{\rho} \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \Big|_{\rho=\theta(\mu, \tau; x, t)} \partial_{\tau} u_k^l(\mu, \tau) d\mu d\tau, \end{aligned}$$

where

$$\alpha_i(x, t) = \text{sgn} \left((a_k - a_j)(\xi, \omega_j(\xi)) \right) \Big|_{\xi=(y_i+z_i)/2}.$$

As j and k are arbitrarily fixed, similarly to the above, we drop the dependence on them in the notation of α_i . Integrating by parts and using (2.28) and (2.29), one gets

$$\begin{aligned} A_0^4(x, t) &= a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \left[c_j(z_i; x, t) p_{jk}(z_i, \omega_j(z_i)) \right. \\ &\quad \times \exp \left\{ \int_x^{z_i} \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \int_1^{z_i} d_k(\mu; z_i, \omega_j(z_i)) u_k^l(\mu, \omega_k(\mu; z_i, \omega_j(z_i))) d\mu \end{aligned}$$

$$\begin{aligned}
& -c_j(y_i; x, t)p_{jk}(y_i, \omega_j(y_i)) \\
& \times \exp \left\{ \int_x^{y_i} \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \int_1^{y_i} d_k(\mu; y_i, \omega_j(y_i)) u_k^l(\mu, \omega_k(\mu; y_i, \omega_j(y_i))) d\mu \Big] \\
& + a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \int_{z_i}^{y_i} c_j(\mu; x, t) d_k(\mu; \mu, \omega_j(\mu)) p_{jk}(\mu, \omega_j(\mu)) \\
& \times \exp \left\{ \int_x^\mu \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} u_k^l(\mu, \omega_k(\mu)) d\mu \\
& + a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \int_{S_i^0(x, t)} \partial_\rho \left[c_j(\rho; x, t) d_k(\mu; \rho, \omega_j(\rho)) p_{jk}(\rho, \omega_j(\rho)) \right. \\
& \times \exp \left\{ \int_x^\rho \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \Big] \exp \left\{ \int_\mu^\rho \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; \mu, \tau)) d\eta \right\} \\
& \times \left(\frac{a_j a_k}{a_k - a_j} \right) (\rho, \omega_j(\rho)) \Big|_{\rho=\theta(\mu, \tau; x, t)} u_k^l(\mu, \tau) d\mu d\tau. \tag{2.30}
\end{aligned}$$

Further we transform the integrals in the last summand making change of variables $(\mu, \tau) \rightarrow (\xi_1, \xi)$ by means of (2.22), taking into account that $\theta(\mu, \tau; x, t) = \xi$ and that the Jacobian of the transformation is given by (2.26), and using the equality

$$\omega_k(\eta; \xi_1, \omega_k(\xi_1; \xi, \omega_j(\xi))) = \omega_k(\eta; \xi, \omega_j(\xi)).$$

We therefore obtain

$$\begin{aligned}
& a_j^{-1}(x, t) \sum_i \int_{y_i}^{z_i} \int_\xi^1 \left(\partial_\xi g_{jk}(\xi; x, t) d_k(\xi_1; \xi, \omega_j(\xi)) + g_{jk}(\xi; x, t) (\partial_\xi d_k)(\xi_1; \xi, \omega_j(\xi)) \right) \\
& \times u_k^l(\xi_1, \omega_k(\xi_1; \xi, \omega_j(\xi))) d\xi_1 d\xi, \tag{2.31}
\end{aligned}$$

where

$$g_{jk}(\xi; x, t) = c_j(\xi; x, t) p_{jk}(\xi, \omega_j(\xi)) \exp \left\{ \int_x^\xi \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\}. \tag{2.32}$$

Set

$$L_g = \max_{\xi \in [0, 1]} \|g_{jk}(\xi; \cdot, \cdot)\|_{\mathcal{C}(\mathbb{R})}, \quad L'_g = \max_{\xi \in [0, 1]} \|\partial_\xi g_{jk}(\xi; \cdot, \cdot)\|_{\mathcal{C}(\mathbb{R})}.$$

Thus, (2.30) and (2.31) entail the estimate

$$\|A_0^4(x, t)\|_{\mathcal{C}(\mathbb{R})} \leq L_a ((2N + 1)L_c L_p L_d L_{exp} + L_d L'_g + L'_d (1 + L_a) L_g) \|u_k^l\|_{\mathcal{C}(\mathbb{R})}. \tag{2.33}$$

The estimation of the first term ($q = 0$) of the series (2.17) is thereby complete.

To handle the second term ($q = 1$), we will concentrate on one arbitrarily fixed integral contributing into $(E_1)_j b_{jk} (C_2 R_2 C_1 R_1 E_2)_k u'''$, namely on

$$A_1(x, t) = \int_0^x d_j(\xi; x, t) b_{jk}(\xi, \omega_j(\xi)) c_k(1; \xi, \omega_j(\xi)) c_r(0; 1, \omega_k(1; \xi, \omega_j(\xi))) \quad (2.34)$$

$$\times \int_0^1 d_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) u_s^l(\xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\xi_1 d\xi$$

for arbitrarily fixed $m+1 \leq s \leq n$ and $1 \leq r \leq m$. Again, to abuse notation, we drop the dependence of A_1 on j, k, s , and m . Similarly to the above, we start with the derivative

$$\partial_x A_1(x, t) = A_1^1(x, t) + A_1^2(x, t) + A_1^3(x, t) + A_1^4(x, t),$$

where

$$A_1^1(x, t) = \frac{b_{jk}(x, t)}{a_j(x, t)} c_k(1; x, t) c_r(0; 1, \omega_k(1))$$

$$\times \int_0^1 d_s(\xi; 0, \omega_r(0; 1, \omega_k(1))) u_s^l(\xi, \omega_s(\xi; 0, \omega_r(0; 1, \omega_k(1)))) d\xi,$$

$$A_1^2(x, t) = \int_0^x \int_0^1 \partial_x [d_j(\xi; x, t) b_{jk}(\xi, \omega_j(\xi)) d_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi))))]$$

$$\times c_k(1; \xi, \omega_j(\xi)) c_r(0; 1, \omega_k(1; \xi, \omega_j(\xi))) u_s^l(\xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\xi_1 d\xi,$$

$$A_1^3(x, t) = \int_0^x \int_0^1 \partial_x [c_k(1; \xi, \omega_j(\xi)) c_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))]$$

$$\times d_j(\xi; x, t) b_{jk}(\xi, \omega_j(\xi)) d_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi))))$$

$$\times u_s^l(\xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\xi_1 d\xi,$$

$$A_1^4(x, t) = -\frac{1}{a_j(x, t)} \int_0^x \int_0^1 d_j(\xi; x, t) b_{jk}(\xi, \omega_j(\xi)) c_k(1; \xi, \omega_j(\xi)) c_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))$$

$$\times d_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi))))$$

$$\times \exp \left\{ \int_0^{\xi_1} \left(\frac{\partial_2 a_s}{a_s^2} \right) (\eta, \omega_s(\eta; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\eta \right\}$$

$$\times \exp \left\{ \int_1^0 \left(\frac{\partial_2 a_r}{a_r^2} \right) (\eta, \omega_r(\eta; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\eta \right\}$$

$$\times \exp \left\{ \int_\xi^1 \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; \xi, \omega_j(\xi))) d\eta \right\} \exp \left\{ \int_x^\xi \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\}$$

$$\times (\partial_2 u_s^l)(\xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\xi_1 d\xi.$$

The following estimates are immediate (in the rest of this section we simply write R in place of $R(a, b)$):

$$\|A_1^1(x, t)\|_{C(\mathbb{R})} \leq RL_a L_b L_d \|u_s^l\|_{C(\mathbb{R})}, \quad (2.35)$$

$$\begin{aligned} & \|A_1^2(x, t)\|_{C(\mathbb{R})} \\ & \leq RL_d \left(\max_{\xi \in [0, 1]} \|\partial_x [d_j(\xi; \cdot, \cdot) b_{jk}(\xi, \omega_j(\xi; \cdot, \cdot))]\|_{C(\mathbb{R})} + L_{exp}^3 L_a L_b L_d' \right) \|u_s^l\|_{C(\mathbb{R})}, \end{aligned} \quad (2.36)$$

$$\|A_1^3(x, t)\|_{C(\mathbb{R})} \leq L_a L_b L_d^2 L_c L_c' R(L_{exp} + L_{exp}^2) \|u_s^l\|_{C(\mathbb{R})}. \quad (2.37)$$

To estimate $A_1^4(x, t)$, we transform the latter similarly to $A_0^4(x, t)$ bringing it to the form where u_s^l does not depend on x and t and no derivatives of u_s^l are involved. All the transformations below will be done again up to the sign.

We first change variables $(\xi_1, \xi) \rightarrow (\mu, \tau)$ by

$$\mu = \xi_1, \quad \tau = \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) \quad (2.38)$$

with the Jacobian $J_1(\mu, \rho) = \tilde{J}_1^{-1}(\xi_1, \xi)|_{(\xi_1, \xi)=(\mu, \rho)}$ where

$$\tilde{J}_1(\xi_1, \xi) = \begin{vmatrix} 1 & 0 \\ \partial_1 \omega_s & \partial_3 \omega_s \partial_3 \omega_r (\partial_2 \omega_k + \partial_3 \omega_k \omega_j'(\xi)) \end{vmatrix}.$$

Here the functions ω_s , ω_r , ω_k , and ω_j are considered having the same arguments as in the formula (2.38). More specifically,

$$\begin{aligned} \tilde{J}_1(\xi_1, \xi) &= \tilde{J}_0(1, \xi) \exp \left\{ \int_0^{\xi_1} \left(\frac{\partial_2 a_s}{a_s^2} \right) (\eta, \omega_s(\eta; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\eta \right\} \\ &\times \exp \left\{ \int_1^0 \left(\frac{\partial_2 a_r}{a_r^2} \right) (\eta, \omega_r(\eta; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\eta \right\}. \end{aligned} \quad (2.39)$$

After the change of the variables, it follows by (9) that

$$\begin{aligned} A_1^4(x, t) &= -a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \int_{S_i^1(x, t)} c_j(\rho; x, t) p_{jk}(\rho, \omega_j(\rho)) \\ &\times d_s(\mu; 0, \omega_r(0; 1, \omega_k(1; \rho, \omega_j(\rho)))) c_k(1; \rho, \omega_j(\rho)) c_r(0; 1, \omega_k(1; \rho, \omega_j(\rho))) \\ &\times \exp \left\{ \int_x^\rho \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \Big|_{\rho=\theta(\mu, \tau; x, t)} \partial_\tau u_s^l(\mu, \tau) d\mu d\tau, \end{aligned} \quad (2.40)$$

where $\theta(\mu, \tau; x, t)$ under the integral over $S_i^1(x, t)$ now denotes the abscissa (in the range (y_i, z_i)) of the point where the characteristics $\omega_j(\rho; x, t)$ and $\omega_k(\rho; 1, \omega_r(1; 0, \omega_s(0; \mu, \tau)))$ intersect. Specifically, $\theta(\mu, \tau; x, t)$ fulfills the equation

$$\omega_j(\theta(\mu, \tau; x, t); x, t) = \omega_k(\theta(\mu, \tau; x, t); 1, \omega_r(1; 0, \omega_s(0; \mu, \tau))). \quad (2.41)$$

Moreover,

$$S_i^1(x, t) = \left\{ (\mu, \tau) \in [0, 1] \times [0, 2\pi] : 0 \leq \mu \leq 1, \right. \\ \left. \omega_s(\mu; \omega_r(0; 1, \omega_k(1; y_i, \omega_j(y_i)))) \leq \tau \leq \omega_s(\mu; \omega_r(0; 1, \omega_k(1; z_i, \omega_j(z_i)))) \right\}. \quad (2.42)$$

Integrating by parts, one gets

$$\begin{aligned} A_1^4(x, t) &= a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \left[c_j(y_i; x, t) p_{jk}(y_i, \omega_j(y_i)) \right. \\ &\quad \times c_k(1; y_i, \omega_j(y_i)) c_r(0; 1, \omega_k(1; y_i, \omega_j(y_i))) \exp \left\{ \int_x^{y_i} \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \\ &\quad \times \int_0^1 d_s(\mu; 0, \omega_r(0; 1, \omega_k(1; y_i, \omega_j(y_i)))) u_s^l(\mu, \omega_s(\mu; 0, \omega_r(0; 1, \omega_k(1; y_i, \omega_j(y_i)))) d\mu \\ &\quad - c_j(z_i; x, t) p_{jk}(z_i, \omega_j(z_i)) c_k(1; z_i, \omega_j(z_i)) c_r(0; 1, \omega_k(1; z_i, \omega_j(z_i))) \\ &\quad \times \exp \left\{ \int_x^{z_i} \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \\ &\quad \times \int_0^1 d_s(\mu; 0, \omega_r(0; 1, \omega_k(1; z_i, \omega_j(z_i)))) u_s^l(\mu, \omega_s(\mu; 0, \omega_r(0; 1, \omega_k(1; z_i, \omega_j(z_i)))) d\mu \Big] \\ &+ a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \int_{S_i^1(x, t)} \partial_\rho \left[c_j(\rho; x, t) d_s(\mu; 0, \omega_r(0; 1, \omega_k(1; \rho, \omega_j(\rho)))) p_{jk}(\rho, \omega_j(\rho)) \right. \\ &\quad c_k(1; \rho, \omega_j(\rho)) c_r(0; 1, \omega_k(1; \rho, \omega_j(\rho))) \exp \left\{ \int_x^\rho \left(\frac{\partial_2 a_j}{a_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \Big] \\ &\quad \times \exp \left\{ \int_1^\rho \left(\frac{\partial_2 a_k}{a_k^2} \right) (\eta, \omega_k(\eta; 1, \omega_r(1; 0, \omega_s(0; \mu, \tau)))) d\eta \right\} \\ &\quad \times \exp \left\{ \int_0^1 \left(\frac{\partial_2 a_r}{a_r^2} \right) (\eta, \omega_r(\eta; 0, \omega_s(0; \mu, \tau))) d\eta \right\} \exp \left\{ \int_\mu^0 \left(\frac{\partial_2 a_s}{a_s^2} \right) (\eta, \omega_s(\eta; \mu, \tau)) d\eta \right\} \\ &\quad \times \left(\frac{a_j a_k}{a_k - a_j} \right) (\rho, \omega_j(\rho)) \Big|_{\rho=\theta(\mu, \tau; x, t)} u_s^l(\mu, \tau) d\mu d\tau. \end{aligned} \quad (2.43)$$

Running our argument further, we change variables $(\mu, \tau) \longrightarrow (\xi_1, \xi)$ under the integrals in the second sum by means of (2.38). Now $\theta(\mu, \tau; x, t) = \xi$ and the Jacobian of the transformation is given by (2.39). Moreover, we take into account that

$$\begin{aligned} \omega_k(\eta; 1, \omega_r(1; 0, \omega_s(0; \xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))))) &= \omega_k(\eta; \xi, \omega_j(\xi)), \\ \omega_r(\eta; 0, \omega_s(0; \xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))))) &= \omega_r(\eta; 1, \omega_k(\eta; \xi, \omega_j(\xi))), \\ \omega_s(0; \xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) &= \omega_s(\eta; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi))). \end{aligned} \quad (2.44)$$

Then the second summand in (2.30) is written as

$$\begin{aligned}
a_j^{-1}(x, t) \sum_i \alpha_i(x, t) \int_{y_i}^{z_i} \int_{\xi}^1 \partial_{\xi} \left[c_j(\xi; x, t) d_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) p_{jk}(\xi, \omega_j(\xi)) \right. \\
\left. \times c_k(1; \xi, \omega_j(\xi)) c_r(0; 1, \omega_k(1; \xi, \omega_j(\xi))) \exp \left\{ \int_x^{\xi} \left(\frac{\partial_2 a_j}{d_j^2} \right) (\eta, \omega_j(\eta)) d\eta \right\} \right] \\
\times u_s^l(\xi_1, \omega_s(\xi_1; 0, \omega_r(0; 1, \omega_k(1; \xi, \omega_j(\xi)))) d\xi_1 d\xi. \quad (2.45)
\end{aligned}$$

On the account of (2.43) and (2.45), the following estimate is fulfilled:

$$\begin{aligned}
\|A_1^4(x, t)\|_{C(\mathbb{R})} \leq L_a R \left(2NL_c L_p L_d L_{exp} + L_c^2 L_d L_p L'_c (L_{exp} + L_{exp}^2) (1 + L_a) \right. \\
\left. + L_d L'_g + L'_d L_{exp}^2 (1 + L_a) L_g \right) \|u_s^l\|_{C(\mathbb{R})}. \quad (2.46)
\end{aligned}$$

Continuing in this fashion, one can easily conclude about the desired estimates for the q -th term of the series (2.17) for $q \geq 1$. More specifically, we have

$$\|A_q^1(x, t)\|_{C(\mathbb{R})} \leq R^q L_a L_b L_d \max_{m+1 \leq s \leq n} \|u_s^l\|_{C(\mathbb{R})}, \quad (2.47)$$

$$\begin{aligned}
\|A_q^2(x, t)\|_{C(\mathbb{R})} \leq R^q L_d \left(\max_{\xi \in [0, 1]} \|\partial_x [d_j(\xi; \cdot, \cdot) b_{jk}(\xi, \omega_j(\xi; \cdot, \cdot))]\|_{C(\mathbb{R})} \right. \\
\left. + L_{exp}^{2q+1} L_a L_b L'_d \right) \max_{m+1 \leq s \leq n} \|u_s^l\|_{C(\mathbb{R})}, \quad (2.48)
\end{aligned}$$

$$\|A_q^3(x, t)\|_{C(\mathbb{R})} \leq L_a L_b L_d^2 L_c L'_c R^q \sum_{i=1}^{2q} L_{exp}^i \max_{m+1 \leq s \leq n} \|u_s^l\|_{C(\mathbb{R})}, \quad (2.49)$$

and

$$\begin{aligned}
\|A_q^4(x, t)\|_{C(\mathbb{R})} \leq L_a R^q \left(2NL_c L_p L_d L_{exp} + L_c^2 L_d L_p L'_c (1 + L_a) \sum_{i=1}^{2q} L_{exp}^i \right. \\
\left. + L_d L'_g + L'_d L_{exp}^{2q} (1 + L_a) L_g \right) \max_{m+1 \leq s \leq n} \|u_s^l\|_{C(\mathbb{R})}. \quad (2.50)
\end{aligned}$$

As $L_{exp} \geq 1$, then

$$\sum_{i=1}^{2q} L_{exp}^i \leq 2q L_{exp}^{2q}.$$

Hence

$$\|A_q^3(x, t)\|_{C(\mathbb{R})} \leq 2L_{exp} L_a L_b L_d^2 L_c L'_c q (R L_{exp}^2)^q \max_{m+1 \leq s \leq n} \|u_s^l\|_{C(\mathbb{R})} \quad (2.51)$$

and

$$\begin{aligned}
\|A_q^4(x, t)\|_{C(\mathbb{R})} \leq \max_{m+1 \leq s \leq n} \|u_s^l\|_{C(\mathbb{R})} L_a \left(2NL_c L_p L_d L_{exp} R^q \right. \\
+ 2L_c^2 L_d L_p L'_c q (R L_{exp}^2)^q (1 + L_a) + L_d L'_g R^q \\
\left. + L'_d (R L_{exp}^2)^q (1 + L_a) L_g \right) \quad (2.52)
\end{aligned}$$

Note that the norm of the x -derivative of the q -th term with $q \geq 1$ in the series (2.17) is estimated from above by

$$\begin{aligned} & L_d \left[L_a L_b + \max_{\xi \in [0,1]} \|\partial_x [d_j(\xi; \cdot, \cdot) b_{jk}(\xi, \omega_j(\xi; \cdot, \cdot))]\|_{\mathcal{C}(\mathbb{R})} \right. \\ & \quad + L_{exp}^{2q+1} L_a L_b L_d' + 2L_a L_b L_d L_c L_c' q L_{exp}^{2q} + 2N L_a L_c L_p L_{exp} + 2L_a L_c^2 L_p L_c' q L_{exp}^{2q} (1 + L_a) \\ & \quad \left. + L_a L_g' + L_d^{-1} L_d' L_{exp}^{2q} (1 + L_a) L_g \right] \left(R \sum_{i=m+1}^n \sum_{s,k=1}^m |r_{sk}^1 r_{ki}^0| \right)^q \max_{m+1 \leq s \leq n} \|u_s^l\|_{\mathcal{C}(\mathbb{R})}. \end{aligned}$$

To finish with the uniform boundedness of the x -derivative, it remains to take into account the contractibility condition (8) and to note that the series

$$\sum_{q=0}^{\infty} \left[q \left(R L_{exp}^2 \sum_{r=m+1}^n \sum_{s,k=1}^m |r_{sk}^1 r_{kr}^0| \right)^q \right] \quad (2.53)$$

is convergent.

The uniform boundedness of the t -derivative of the series follows easily from the uniform boundedness of the x -derivative we just proved and from the following equalities:

$$\partial_t [(A_q u_s)(x, t)] = a_j(x, t) \partial_x [(A_q u_s)(x, t)] - A_q^1(x, t), \quad q \in \mathbb{N}_0. \quad (2.54)$$

Summarizing, we obtain the following estimate:

$$\|\nabla_{(x,t)} [(A u^l)(x, t)]\|_{\mathcal{C}(\mathbb{R}^n)} \leq C \|u^l\|_{\mathcal{C}(\mathbb{R}^n)} \leq C (\|u\|_{\mathcal{C}(\mathbb{R}^n)} + \|u_s^l - u_s\|_{\mathcal{C}(\mathbb{R}^n)}),$$

the constant C being independent of $(x, t) \in [0, 1] \times [0, 2\pi]$, $u \in N$, and $l \in \mathbb{N}$. By (2.14), the constant C can be chosen so that

$$\|\nabla_{(x,t)} [(A u^l)(x, t)]\|_{\mathcal{C}(\mathbb{R}^n)} \leq C \|u\|_{\mathcal{C}(\mathbb{R})}.$$

It remains to recall that u varies in N , which is a bounded domain of $\mathcal{C}(\mathbb{R}^n)$. The proof of Items (i) and (ii) of Theorem 1.2 is thereby complete.

If $b_{jk} = 0$ for all $j \neq k$, then the operators B_1 and B_2 and, hence, the operator K vanish. This entails Assertion (iii).

3 Fredholm alternative (proof of Theorem 1.3)

We first introduce some more notation. Define linear operators $\tilde{C}_1, \tilde{E}_1 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^m))$, $\tilde{C}_2, \tilde{E}_2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}))$, $(\tilde{B}_1, \tilde{B}_2) \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$, $\tilde{R}_1 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^{n-m}), \mathcal{C}(\mathbb{R}^m))$, and $\tilde{R}_2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^m), \mathcal{C}(\mathbb{R}^{n-m}))$ by

$$\begin{aligned} \left(\tilde{C}_1 u^1 \right) (x, t) &= (\tilde{c}_1(0; x, t) u_1(x, t), \dots, \tilde{c}_m(0; x, t) u_m(x, t)) \\ \left(\tilde{C}_2 u^2 \right) (x, t) &= (\tilde{c}_{m+1}(1; x, t) u_{m+1}(x, t), \dots, \tilde{c}_n(1; x, t) u_n(x, t)) \end{aligned}$$

$$\begin{aligned}
(\tilde{R}_1 u^2)(x, t) &= \left[- \sum_{k=m+1}^n r_{kj}^0 \frac{a_k(0, t)}{a_j(0, t)} u_k(0, \omega_j(0)) \right]_{j=1}^m, \\
(\tilde{R}_2 u^1)(x, t) &= \left[- \sum_{k=1}^m r_{kj}^1 \frac{a_k(1, t)}{a_j(1, t)} u_k(1, \omega_j(1)) \right]_{j=m+1}^n, \\
(\tilde{B}_1 u)(x, t) &= \left[\sum_{k \neq j} b_{kj} u_k \right]_{j=1}^m, \\
(\tilde{B}_2 u)(x, t) &= \left[\sum_{k \neq j} b_{kj} u_k \right]_{j=m+1}^n, \\
(\tilde{E}_1 u^1)(x, t) &= \left[\int_0^x \tilde{d}_j(\xi; x, t) u_j(\xi, \omega_j(\xi)) d\xi \right]_{j=1}^m, \\
(\tilde{E}_2 u^2)(x, t) &= \left[\int_x^1 \tilde{d}_j(\xi; x, t) u_j(\xi, \omega_j(\xi)) d\xi \right]_{j=m+1}^n,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{c}_j(\xi; x, t) &= \exp \int_x^\xi \left(\frac{\partial_x a_j - b_{jj}}{a_j} \right) (\eta, \omega_j(\eta; x, t)) d\eta, \\
\tilde{d}_j(\xi; x, t) &= \frac{\tilde{c}_j(\xi; x, t)}{a_j(\xi, \omega_j(\xi; x, t))}.
\end{aligned} \tag{3.1}$$

In these notations, all continuous solutions to the adjoint problem (11)–(13) satisfy the following operator equation:

$$u = \tilde{M}u + \tilde{K}u,$$

where

$$\begin{aligned}
\tilde{M}u &= (\tilde{C}_1 \tilde{R}_1 u^2, \tilde{C}_2 \tilde{R}_2 \tilde{C}_1 \tilde{R}_1 u^2), \\
\tilde{K}u &= (-\tilde{E}_1 \tilde{B}_1 u, \tilde{E}_2 \tilde{B}_2 u - \tilde{C}_2 \tilde{R}_2 \tilde{E}_1 \tilde{B}_1 u).
\end{aligned}$$

In Section 2 we proved the Fredholm alternative for $I - M - K \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ in the following abstract form:

$$\dim \ker(I - M - K) = \dim \ker(I - M - K)^* < \infty, \quad \left. \begin{aligned} \text{im}(I - M - K) = \{ f \in \mathcal{C}(\mathbb{R}^n) : [u, f]_{\mathcal{C}(\mathbb{R}^n)} = 0 \text{ for all } u \in \ker(I - M - K)^* \} \end{aligned} \right\}. \tag{3.2}$$

Here $(I - M - K)^*$ is the dual operator to $I - M - K$, i.e. a linear bounded operator from $(\mathcal{C}(\mathbb{R}^n))^*$ into $(\mathcal{C}(\mathbb{R}^n))^*$, and $[\cdot, \cdot]_{\mathcal{C}(\mathbb{R}^n)} : (\mathcal{C}(\mathbb{R}^n))^* \times \mathcal{C}(\mathbb{R}^n) \rightarrow \mathbb{R}$ is the dual pairing in $\mathcal{C}(\mathbb{R}^n)$.

To prove Items (i) and (ii) of Theorem 1.3, we need a bit more than (3.2), namely

$$\text{im}(I - M - K) = \left\{ f \in \mathcal{C}(\mathbb{R}^n) : \langle f, u \rangle_{L^2} = 0 \text{ for all } u \in \ker(I - \tilde{M} - \tilde{K}) \right\}.$$

Directly from the definitions of the operators M , \tilde{M} , K , and \tilde{K} it follows that

$$\langle (I - M - K)u, \tilde{u} \rangle_{L^2} = \langle u, (I - \tilde{M} + \tilde{K})\tilde{u} \rangle_{L^2} \text{ for all } u, \tilde{u} \in \mathcal{C}(\mathbb{R}^n). \quad (3.3)$$

Using the continuous dense embedding $\mathcal{C}(\mathbb{R}^n) \hookrightarrow L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n) \hookrightarrow (\mathcal{C}(\mathbb{R}^n))^*$, it makes sense to compare the subspaces $\ker(I - M - K)^*$ of $(\mathcal{C}(\mathbb{R}^n))^*$ and $\ker(I - \tilde{M} - \tilde{K})$ of $\mathcal{C}(\mathbb{R}^n)$:

Lemma 3.1 $\ker(I - M - K)^* = \ker(I - \tilde{M} - \tilde{K})$.

Proof. For all $u, \tilde{u} \in \mathcal{C}(\mathbb{R}^n)$ we have

$$\begin{aligned} \left\langle \left(I - \tilde{M} - \tilde{K} \right) \tilde{u}, u \right\rangle_{L^2} &= \langle \tilde{u}, (I - M - K)u \rangle_{L^2} \\ &= [\tilde{u}, (I - M - K)u]_{\mathcal{C}(\mathbb{R}^n)} = [(I - M - K)^* \tilde{u}, u]_{\mathcal{C}(\mathbb{R}^n)}. \end{aligned}$$

This implies $\ker(I - \tilde{M} - \tilde{K}) \subseteq \ker(I - M - K)^*$.

Now, take an arbitrary $u \in \ker(I - M - K)^*$ and show that $u \in \ker(I - \tilde{M} - \tilde{K})$. Take a sequence $u^l \in \mathcal{C}(\mathbb{R}^n)$ such that

$$u^l \rightarrow u \text{ in } (\mathcal{C}(\mathbb{R}^n))^*. \quad (3.4)$$

Then for all $v \in \mathcal{C}(\mathbb{R}^n)$ we have

$$\begin{aligned} 0 &= [(I - M - K)^* u, v]_{\mathcal{C}(\mathbb{R}^n)} = [u, (I - M - K)u]_{\mathcal{C}(\mathbb{R}^n)} \\ &= \lim_{l \rightarrow \infty} [u^l, (I - M - K)v]_{\mathcal{C}(\mathbb{R}^n)} = \lim_{l \rightarrow \infty} \left\langle \left(I - \tilde{M} - \tilde{K} \right) u^l, v \right\rangle_{L^2}. \end{aligned}$$

It follows that

$$\left(I - \tilde{M} - \tilde{K} \right) u^l \rightharpoonup 0 \text{ in } L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n). \quad (3.5)$$

Further we use the bijectivity of the operator $I - \tilde{M} \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$, which follows from the proof of the bijectivity of the operator $I - M \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ and condition (14). We therefore have

$$\left(I - \left[\left(I - \tilde{M} \right)^{-1} \tilde{K} \right]^2 \right) u^l \rightharpoonup 0 \text{ in } L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n). \quad (3.6)$$

Let us distinguish two cases.

Case 1. The sequence $\|u^l\|_{\mathcal{C}(\mathbb{R}^n)}$ is bounded. Since for the problem (11)–(13) all conditions of Theorem 1.2 are fulfilled, the operator $\left[\left(I - \tilde{M} \right)^{-1} \tilde{K} \right]^2 \in \mathcal{L}(\mathcal{C}(\mathbb{R}^n))$ is compact as follows from the proof in Section 2.3. Then there is a subsequence of u^l (let us keep the same notation u^l for the subsequence) such that $\left[\left(I - \tilde{M} \right)^{-1} \tilde{K} \right]^2 u^l$ converges in $\mathcal{C}(\mathbb{R}^n)$ to a function $v \in \mathcal{C}(\mathbb{R}^n)$. Hence (3.6) implies the weak convergence

$$u^l \rightharpoonup v \text{ in } L^2((0, 1) \times (0, 2\pi); \mathbb{R}^n). \quad (3.7)$$

By (3.4), we obtain the equality $v = u$. It follows by (3.5) and (3.7) that

$$(I - \tilde{M} - \tilde{K})v = 0 \text{ a.e. on } (0, 1) \times (0, 2\pi).$$

The left-hand side is a continuous vector-function that equals zero almost everywhere on $(0, 1) \times (0, 2\pi)$. Hence it equals zero everywhere on $(0, 1) \times (0, 2\pi)$. This means that $u \in \ker(I - \tilde{M} - \tilde{K})$.

Case 2. The sequence $\|u^l\|_{\mathcal{C}(\mathbb{R}^n)}$ is unbounded. Then there is a subsequence, denoted again by u^l , such that $\|u^l\|_{\mathcal{C}(\mathbb{R}^n)} \geq c$ for some $c > 0$ and all $l \in \mathbb{N}$. Consider a normalized sequence $u^l/\|u^l\|_{\mathcal{C}(\mathbb{R}^n)}$. Applying now the argument used in Case 1, we conclude that $u^l/\|u^l\|_{\mathcal{C}(\mathbb{R}^n)}$ converges in $\mathcal{C}(\mathbb{R}^n)$ as $l \rightarrow \infty$ to a function v with $\|v\|_{\mathcal{C}(\mathbb{R}^n)} = 1$. From the other side, $u^l/\|u^l\|_{\mathcal{C}(\mathbb{R}^n)} \rightarrow 0$ in $(\mathcal{C}(\mathbb{R}^n))^*$ as $l \rightarrow \infty$, what follows from (3.4) and the unboundedness of $\|u^l\|_{\mathcal{C}(\mathbb{R}^n)}$. We get $v = 0$, a contradiction with $\|v\|_{\mathcal{C}(\mathbb{R}^n)} = 1$. Hence Case 2 is impossible and the proof is complete. \square

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